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CONSTRUCTION OF THE FUNDAMENTAL SOLUTION OF A CLASS OF DEGENERATE PARABOLIC EQUATIONS OF HIGH ORDER

In the article, using the modified Levy method, a Green’s function for a class of ultraparabolic equations of high order with an arbitrary number of parabolic degeneration groups is constructed. The modified Levy method is developed for high-order Kolmogorov equations with coefficients depending on all variables, while the frozen point, which is a parametrix, is chosen so that an exponential estimate of the fundamental solution and its derivatives is conveniently used.

Key words and phrases: degenerated parabolic equations, modified Levy’s method, Kolmogorov’s equation, fundamental solution, parametrix.

INTRODUCTION

A fundamental solution of the inverse Cauchy problem for degenerate parabolic equations of second-order variables with smooth coefficients was constructed first by M. Weber [10]. Under the same conditions on the coefficients, a fundamental solution of the Cauchy problem was constructed in [5], in the case of Holder coefficients for second-order equations with two degenerate groups. The Levy method was modified in [7], and in Banach spaces in [8], for the second order Kolmogorov systems with one degeneracy group [4]. The Kolmogorov equation of high order has features that make it easy to use the Levy method for constructing a fundamental solution. The parametric method was applied to a degenerate parabolic equation of high order with one group of parabolic degeneracy variables in [2, 3, 9] and with two degenerate groups in [1] and with four degenerate groups for Kolmogorov type systems of the second order in [6]. We modified the Levy method with respect to the properties of a fundamental solution of high-order Kolmogorov-type equations with coefficients dependent only on $t$, in particular a selected point which is a parameter so that an exponential estimate of the fundamental solution and its derivatives is conveniently used.

1 DESIGNATION, TASK STATEMENT AND MAIN RESULTS

Let us denote by $n_j \in \mathbb{N}, j = 1, p, n_1 \geq n_2 \geq \ldots \geq n_p, n_0 = \sum_{j=1}^{p} n_j, x = (x_1, \ldots, x_p), x_j = (x_{j1}, \ldots, x_{jn_j}), x \in R^{n_0}, x_j \in R^{n_j}, \xi = (\xi_1, \ldots, \xi_p), \xi_j = (\xi_{j1}, \ldots, \xi_{jn_j}), \xi_j \in R^{n_j}, \xi \in R^{n_0}$.
We investigate the Cauchy problem for the equation

\[ x^{(j)} = (x_1, \ldots, x_j) \in \mathbb{R}, \quad \xi^{(j)} = (\xi_1, \ldots, \xi_j) \in \mathbb{R}, j = \sum p. \] 

\( \Gamma(\alpha) \) is Euler's gamma function and \( B(a, b) \) is Euler's beta function.

\[ \rho_1(t, x_1, \tau, \xi_1) = (|x_1 - \xi_1| (t - \tau)^{-\frac{1}{2p}})^q, \quad q = \frac{2b}{2b - 1}, \quad b \in \mathbb{N}, \]

\[ \rho_j(t, x^{(j)}, \tau, \xi^{(j)}) = \left( |x_j - \xi_j| (t - \tau)^{-\frac{j}{2p}} \right)^q, \quad j = \sum p, \]

\[ \xi(t, \tau) = \left( \xi_1, \xi_2 - \xi_1 (t - \tau), \ldots, \xi_p + \sum_{k=1}^{p-1} (-1)^{p-k} \xi_k (t - \tau)^{p-k} (p-k)! \right). \]

We investigate the Cauchy problem for the equation

\[ \partial_t u \, (t, x) - \sum_{j=1}^{p-1} \sum_{\mu=1}^{j+1} x_j \partial_{x_j+1} u \, (t, x) = \sum_{|k| \leq 2b} a_k \, (t, x) D^k_{x_1} u \, (t, x), \]

with the initial condition

\[ u \, (t, x) \bigg|_{t = \tau} = u_0 \, (x), \quad 0 \leq \tau \leq t \leq T, \]

where \( \tau \) is a fixed number, and operator

\[ \partial_t - \sum_{|k| \leq 2b} a_k \, (t, x) D^k_{x_1}, \quad D^k_{x_1} = \frac{(-1)^k \partial^{k_1 + \cdots + k_n}}{\partial x_1^{k_1} \cdots x_{n_1}^{k_n}}, \quad |k| = k_1 + \cdots + k_{n_1}, \]

is uniformly parabolic in the sense of Petrovsky in the strip \( \Pi_{[0, T]} = (t, x) \in \mathbb{R}^n, 0 \leq t \leq T. \)

Let us suppose that

1) \( a_k \, (t, x), \partial_x a_k \, (t, x), j = \sum p, \) are continuous and bounded in \( \Pi_{[0, T]}, \)

2) there are constants \( a \in (0, 1], r \in (0, 1], \) such that for any \( x \in \mathbb{R}^n, \xi \in \mathbb{R}^n \) and \( t \in [0, T] \)

\[ |a_k \, (t, x) - a_k \, (t, \xi)| \leq c_1 \left( |x_1 - \xi_1|^{a} + \sum_{j=2}^{p} |x_j - \xi_j| \right), \]

\[ \left| \partial_{x_j} a_k \, (t, x) - \partial_{x_j} a_k \, (t, \xi) \right| \leq c_1 |x - \xi|^r, \quad j = \sum p. \]

**Theorem 1.** If conditions 1)–2) are satisfied, then equation (1) has a fundamental solution of the Cauchy problem (1)–(2) \( Z(t, x; \tau, \xi) \) at \( t > \tau \) and the following estimations hold:

\[ \left| \partial_{x_i} Z(t, x; \tau, \xi) \right| \leq A(t - \tau) \sum_{s=1}^{p} \frac{2b(s-1)+1}{2b} (n_s + |m_s|) \Phi (t, x; \tau, \xi), \]

\[ m_s = 0, \quad \text{at } s \neq j, \quad m_j = 1, \quad j = \sum p, \]

\[ \left| \partial_{x_1}^{m_1} Z(t, x; \tau, \xi) \right| \leq A_{m_1} \left( t - \tau \right) \sum_{s=1}^{p} \frac{2b(s-1)+1}{2b} n_s \Phi (t, x; \tau, \xi), \]
\[ |m_1| \leq 2b, \quad x \in \mathbb{R}^{n_0}, \quad \zeta \in \mathbb{R}^{n_0}, \quad 0 \leq t < T, \text{ where} \]

\[
\Phi(t, x; \tau, \zeta) = \sum_{j=1}^{\infty} A^j \Gamma \left( 1 + \frac{s \alpha^*}{2b} \right) \Gamma \left( \frac{\alpha^*}{2b} \right) \Gamma^{-1} \left( 1 + \frac{\alpha^* (1 + s)}{2b} \right) \times \exp \left\{ -c_0 \rho_1(t, x_1, \tau, \zeta_1) - 2^{-2sp} c_0 \sum_{j=2}^{p} \rho_j(t, x, t, \tau, \zeta^{(j)}) \right\},
\]

and positive constants \( A, A_{m_1}, c_0 \) depend on \( n_0, 2b, c_1, \alpha, r, \) and the constant of parabolicity of the operator (3) is \( \sup_{(t,x) \in [1, T]} |a_k(t, x)| \) and \( \alpha^* = \min(\alpha, r). \)

**Proof.** To prove the theorem, we write equation (1) in the form

\[
\partial_t u(t, x) - \sum_{j=1}^{p} \sum_{\mu=1}^{n_j+1} x_{j\mu} \partial_{x_j+1, \mu} u(t, x) = \sum_{|k|=2b} a_k(t, \zeta(t, \tau)) D^k_{x_1} u(t, x)
\]

\[ + \sum_{|k|=2b} [a_k(t, x) - a_k(t, \zeta(t, \tau))] D^k_{x_1} u(t, x) + \sum_{|k|<2b} a_k(t, x) D^k_{x_1} u(t, x). \tag{4} \]

Let us denote by \( Z_0(t, x; \tau, \zeta; \zeta(t, \tau)) \) the fundamental solution of equation

\[
\partial_t u(t, x) - \sum_{j=1}^{n_j+1} \sum_{\mu=1}^{n_j} x_{j\mu} \partial_{x_j+1, \mu} u(t, x) = \sum_{|k|=2b} a_k(t, \zeta(t, \tau)) D^k_{x_1} u(t, x). \tag{5} \]

Fundamental solution \( Z_0(t, x; \tau, \zeta; \zeta(t, \tau)) \) of equation (5) is constructed in [5], where \( \zeta \) is fixed. For derivatives of \( Z_0(t, x; \tau, \zeta; \zeta(t, \tau)) \) the following inequalities are performed

\[
|\partial_x^n Z_0(t, x; \tau, \zeta; \zeta(t, \tau))| \leq C_m (t - \tau)^{-\frac{2^{p(n+1)} |m|}{2b} - \sum_{i=1}^{p} \frac{2^{b(n+1)} |m_i|}{2b}} \times \exp \left\{ -c_0 \left( \sum_{j=2}^{p} \rho_j(t, x, t, \tau, \zeta^{(j)}) + \rho_1(t, x_1, \tau, \zeta_1) \right) \right\}, \tag{6} \]

where \( |m| = \sum_{j=1}^{p} |m_j|, \quad |m_j| = \sum_{k=1}^{n_j} m_{jk}, \quad t > \tau, \quad C_m > 0. \)

Fundamental solution \( Z(t, x; \tau, \zeta) \) of equation (1) will be sought in the form

\[
Z(t, x; \tau, \zeta) = Z_0(t, x; \tau, \zeta; \zeta(t, \tau)) + \int_{\tau}^{t} \frac{d\beta}{\int_{\mathbb{R}^{n_0}}} Z_0(t, x; \beta, \gamma; \gamma(t, \beta)) \varphi(\beta, \gamma; \tau, \zeta) d\gamma, \tag{7} \]

where \( \varphi(t, x; \tau, \zeta) \) is an unknown absolutely integrable on \( \mathbb{R}^{n_0} \) function at \( t > \tau. \)

We substitute (6) into equation (1) with respect to the function \( \varphi(t, x; \tau, \zeta), \) then

\[
\varphi(t, x; \tau, \zeta) = K(t, x; \tau, \zeta) + \int_{\tau}^{t} K(t, x; \beta, \gamma) \varphi(\beta, \gamma; \tau, \zeta) d\gamma, \tag{8} \]

where

\[
K(t, x; \tau, \zeta) = \sum_{|k|=2b} (a_k(t, x) - a_k(t, \zeta(t, \tau))) D^k_{x_1} Z_0(t, x; \tau, \zeta; \zeta(t, \tau)) + \sum_{|k|<2b} a_k(t, x) D^k_{x_1} Z_0(t, x; \tau, \zeta; \zeta(t, \tau)). \tag{9} \]
The solution of equation (8) can be represented by a Neumann series

\[ \varphi(t, x; \tau, \zeta) = \sum_{n=1}^{\infty} K_n(t, x; \tau, \zeta), \]  

where

\[ K(t, x; \tau, \zeta) = K_1(t, x; \tau, \zeta); \quad K_n(t, x; \tau, \zeta) = \int_{\tau}^{t} d\beta \int_{\mathbb{R}^n_0} K(t, x; \beta, \gamma) K_{n-1}(\beta, \gamma; \tau, \zeta) d\gamma. \]  

Lemma 1.

For any points \((t, x), (\beta, \zeta), (\tau, y), 0 \leq \tau < \beta < t, x \in \mathbb{R}^n_0, \zeta \in \mathbb{R}^n_0, y \in \mathbb{R}^n_0, b \in \mathbb{N}, 2b > 2\) the following inequality holds

\[ \rho_1(t, x_1, \beta, \xi_1) + \sum_{j=2}^{p} \rho_j(t, x(j), \beta, \xi(j)) + \rho_1(\beta, \xi_1, \tau, y_1) + \sum_{j=2}^{p} \rho_j(\beta, \xi(j), \tau, y(j)) \geq 2^{-2p} \left( \sum_{j=2}^{p} \rho_j(t, x(j), \tau, y(j)) + \rho_1(t, x_1, \tau, y_1) \right). \]  

(11)

The proof of Lemma 1 is based on the inequalities

\[ \rho_p(t, x^{(p)}, \beta, \xi^{(p)}) + \rho_p(\beta, \xi^{(p)}, \tau, y^{(p)}) \geq 2^{-2} \left( |x_p - y_p| + \sum_{j=1}^{p-1} \left| x_k(t - \beta)^{p-k} + \xi_k(\beta - \tau)^{p-k} \right| \frac{1}{(p-k)!} (t - \tau)^{-p+1-\frac{1}{2b}} \right)^q. \]  

(12)

From (12) we can get

\[ \left( |x_p - y_p| + \sum_{k=1}^{p-1} \left[ x_k(t - \beta)^{p-k} + \xi_k(\beta - \tau)^{p-k} \right] ((p-k)!)^{-1} (t - \tau)^{-p+1-\frac{1}{2b}} \right)^q \geq 2^{-2} \left( |x_p - y_p| + \sum_{k=1}^{p-1} \left[ x_k(t - \beta)^{p-k} + \xi_k(\beta - \tau)^{p-k} \right] ((p-k)!)^{-1} \right. \]

\[ \times \left. \frac{x_1((\beta - \tau)^{p-1} + (t - \beta)^{p-1})}{(p-1)!} (t - \tau)^{-p+1-\frac{1}{2b}} \right)^q \]

\[ - \sum_{\mu=1}^{n_p} \left| x_{1\mu} - \xi_{1\mu} \right| ((\beta - \tau)^{p-1}(p-k)!)^{-1} (t - \tau)^{-p+1-\frac{1}{2b}} \right)^q. \]  

(13)

Applying (12) to the first part of (13) \((p - 2)\) times, we have

\[ \left( |x_p - y_p| + \sum_{k=1}^{p-1} (x_k(t - \beta)^{p-k} + \xi_k(\beta - \tau)^{p-k}) ((p-k)!)^{-1} (t - \tau)^{-p+1-\frac{1}{2b}} \right)^q \]  

\[ - \sum_{\mu=1}^{n_p} \left| x_{1\mu} - \xi_{1\mu} \right| ((\beta - \tau)^{p-1}(p-k)!)^{-1} (t - \tau)^{-p+1-\frac{1}{2b}} \right)^q. \]
We will collect all of the terms that contain $\xi_{p-1}$.

\[
\begin{align*}
\geq & \ 2^{-2(p-1)} \rho_p \left( t, x^{(p)}, \beta, \xi^{(p)} \right) - \sum_{\mu=1}^{n_p} \left( x_{1\mu} - \xi_{1\mu} \right) (\beta - (p - 1))^{-1}(t - \tau)^{-p+1-\frac{1}{2}} \\
& - \sum_{j=2}^{p-1} \sum_{\mu=1}^{n_p} 2^{-2(j-1)} \left( x_{1\mu} - \xi_{1\mu} + \sum_{k=2}^{j-1} x_{k\mu} (t - \beta)^{j-k} ((j - k)!)^{-1} \left( \frac{p - j}{(p - 1)!} \right) (t - \tau)^{-p+1-\frac{1}{2}} \right) \cdot \end{align*}
\]

Taking into account the inequalities (11)–(14), we get

\[
\begin{align*}
\rho_p \left( t, x^{(p)}, \beta, \xi^{(p)} \right) + \rho_p \left( \beta, \xi^{(p)}, \tau, y^{(p)} \right) & \geq 2^{-2p} \rho_p \left( t, x^{(p)}, \tau, y^{(p)} \right) \\
& - \sum_{j=2}^{p-1} \sum_{\mu=1}^{n_p} 2^{-2(j-1)} \left( x_{1\mu} - \xi_{1\mu} + \sum_{k=2}^{j-1} x_{k\mu} (t - \beta)^{j-k} ((j - k)!)^{-1} \left( \frac{p - j}{(p - 1)!} \right) (t - \tau)^{-p+1-\frac{1}{2}} \right) \\
& \times (t - \tau)^{-p+1-\frac{1}{2}} \cdot \end{align*}
\]

We will collect all of the terms that contain $x_{p-1} - \xi_{p-1}$

\[
\begin{align*}
\rho_{p-1} \left( \beta, x^{(p-1)}, \beta, \xi^{(p-1)} \right) - 2^{-2(p-1)} \sum_{\mu=1}^{n_p} \left( x_{p1\mu} - \xi_{p1\mu} + \sum_{k=1}^{p-2} x_{k\mu} (t - \beta)^{p-1-k} \right) \\
& \times \frac{1}{((p - 1 - k)!)^q} (\beta - (p - 1 - k - \frac{1}{2}))^{-1} \geq \sum_{\mu=1}^{n_p} \left( x_{p1\mu} - \xi_{p1\mu} \right) \\
& + \sum_{k=1}^{p-2} x_{k\mu} \frac{(t - \beta)^{p-1-k}}{(p - 1 - k)!} (t - \beta)^{-p+1-\frac{1}{2}} \right) \right) \cdot \end{align*}
\]

Repeating all inequalities (12), (16) for the terms $\rho_j \left( t, x^{(j)}, \beta, \xi^{(j)} \right) + \rho_j \left( \beta, \xi^{(j)}, \tau, y^{(j)} \right)$, $j = 1, p - 1$, and adding their together we have

\[
\begin{align*}
\rho_1 \left( t, x_1, \beta, \xi_1 \right) + \rho_1 \left( \beta, \xi_1, \tau, y_1 \right) + \sum_{j=2}^{p} \rho_j \left( t, x^{(j)}, \beta, \xi^{(j)} \right) + \rho_j \left( \beta, \xi^{(j)}, \tau, y^{(j)} \right) \\
& \geq 2^{-2p} \left( \sum_{j=2}^{p} \rho_j \left( t, x^{(j)}, \tau, y^{(j)} \right) + \rho_1 \left( t, x_1, \tau, y_1 \right) \right) \cdot \end{align*}
\]

Lemma 2. The following estimations are performed for reproducing kernels:

\[
\begin{align*}
|K_m \left( t, x; \beta, \xi \right)| & \leq A_m^m(t - \tau) - \frac{\left( 1 + 2b(j-1) \right) m_n \cdot (m + 1) + m^2}{2b} \\
& \times \exp \left\{ \rho_1 \left( t, x_1, \beta, \xi_1 \right) - 2^{-2p} \frac{\sum_{j=2}^{p} \rho_j \left( t, x^{(j)}, \beta, \xi^{(j)} \right)} \right\} \cdot \end{align*}
\]

at $m \leq m^* = \left[ \sum_{j=1}^{p} \frac{(1 + 2b(j-1)) m_n + 2b}{b} \right] + 1$;
To prove the existence of derivatives \( m \) at \( \partial \xi \), then there are continuous derivatives by

\[
\sum \lim_{j \to 0} \rho_j (t, x, \tau, \xi(j))
\]

From (17), (18) it follows the convergence of a series (9) following for \( \| \phi (t, x; \tau, \xi) \|
\]

Let us prove the existence of derivatives \( \partial_\tau \xi \) \( j = \sum p \), at \( t > \tau \).

Under the assumption 1), there are continuous derivatives \( \partial_\xi \xi \) \( j = \sum p \) satisfying the estimations

\[
\left| \partial_\xi \xi (t, x; \tau, \xi) \right| \leq A \exp \left\{-c \left( \sum_{j=2}^{p} \rho_j (t, x, \tau, \xi(j)) + \rho_1 (t, x, \tau, \xi_1) \right) \right\}
\]

To prove the existence of derivatives \( \partial_\xi \xi \) \( j = \sum p \) we use the following property of the fundamental solution of equation (5) with \( \xi (t, \tau) = y \), where \( y \) is a parameter

\[
\partial_\xi \xi \xi (t, x; \tau, \xi) \leq C_m \exp \left\{-c \left( \sum_{j=2}^{p} \rho_j (t, x, \tau, \xi(j)) + \rho_1 (t, x, \tau, \xi_1) \right) \right\}
\]

Let us consider \( \partial_\xi \xi \xi (t, x; \beta, \gamma, \mu = \frac{1}{n_p} \). Then

\[
\partial_\xi \xi \xi (t, x; \beta, \gamma, \mu = \frac{1}{n_p} \]

\[
\sum_{|k| = 2b} (\partial_\xi \xi \xi a_k (t, x) - \partial_\xi \xi \xi a_k (t, \gamma (t, \beta)))
\]

\[
\times D_k \xi \xi \xi (t, x; \beta, \gamma, \gamma (t, \beta)) + \sum_{|k| = 2b} (\partial_\xi \xi \xi a_k (t, \gamma (t, \beta))) \xi \xi \xi (t, x; \beta, \gamma, \gamma (t, \beta))
\]

\[
+ \sum_{|k| = 2b} (a_k (t, x) - \partial_\xi \xi \xi a_k (t, \beta)) \partial_\xi \xi \xi (t, x; \beta, \gamma, \gamma (t, \beta))
\]

\[
+ \sum_{|k| < 2b} D_k \xi \xi \xi (t, x; \beta, \gamma, \gamma (t, \beta)) .
\]
Let us rewrite (22) by a convenient form for applications
\[
\partial_{x_{pp}} K(t, x; \beta, \gamma) = \sum_{|k|=2b} (\partial_{x_{pp}} a_k(t, x) - \partial_{\gamma_{pp}} a_k(t, \gamma(t, \beta)))
\]
\[
\times D_k^p Z_0(t, x; \beta, \gamma; \gamma(t, \beta)) - \partial_{\gamma_{pp}} \left( \sum_{|k|=2b} (a_k(t, x) - a_k(t, \gamma(t, \beta))) \right)
\]
\[
\times D_k^p Z_0(t, x; \beta, \gamma; \gamma(t, \beta)) + \sum_{|k|=2b} (a_k(t, x) - a_k(t, \gamma(t, \beta)))
\]
\[
\times D_k^p \partial_{\gamma_{pp}} D_k^p Z_0(t, x; \beta, \gamma; \gamma(t, \beta)) + \sum_{|k|<2b} (a_k(t, x) \partial_{\gamma_{pp}} D_k^p Z_0(t, x; \beta, \gamma; \gamma(t, \beta))
\]
\[
+ \sum_{|k|<2b} a_i \partial_{\gamma_{pp}} D_k^p Z_0(t, x; \beta, \gamma; \gamma(t, \beta)) \bigg|_{\gamma=\gamma} - \sum_{|k|<2b} a_k(t, x) D_k^p \partial_{\gamma_{pp}} Z_0(t, x; \beta, \gamma; \gamma(t, \beta))
\]
where \( \mu = \frac{1}{1, n_p}, \gamma = (\gamma_1, \ldots, \gamma_{p-1}, \gamma_p) \). Using the images (23), estimates (6) and (21) and integrating by parts of the terms with \( \partial_{\gamma_{pp}} \), we can get \( \partial_{x_{pp}} K_2(t, x; \tau, \xi) = \lim_{h \to 0} \int_{|t-h|}^{t-h} d\beta \int_{R_0} \partial_{x_{pp}} K(t, x; \beta, \gamma) \bigg|_{\gamma=\gamma} K(\beta, \gamma; \tau, \xi) d\gamma \).

From the estimations of reproducing kernel (18), estimations of derivatives of the kernel (20) and Lemma 1, we obtain
\[
\left| \partial_{x_{pp}} K_2(t, x; \tau, \xi) \right| \leq A_2 \exp \left\{ -c_2 \left( 1 - \frac{s}{s} \right) \rho_1(t, x_1, \tau, \xi_1) \right\} + 2^{2-2p} \sum_{j=1}^p \rho_j (t, x_{(j)}, \tau, \xi_{(j)}) + \rho_1(t, x_1, \tau, \xi_1) \right\} (t - \tau) - \sum_{j=1}^{1, n_p} \left( 1 + (2h-j) \right) - \frac{p - (1 - \alpha) / 2b}{2b}
\]
\[
+ \rho_1(t, x_1, \tau, \xi_1) \right\} (t - \tau) - \sum_{j=1}^{1, n_p} \left( 1 + (2h-j) \right) - \frac{p - (1 - \alpha) / 2b}{2b}, \quad \mu = \frac{1}{1, n_p},
\]
Taking into account the estimation (24), we can estimate the series \( \sum_{m=1}^{\infty} \partial_{x_{pp}} K_m(t, x; \tau, \xi) \) by a converging series:
\[
\sum_{m=1}^{\infty} \partial_{x_{pp}} K_m(t, x; \tau, \xi) \leq \sum_{m=1}^{l-1} A_m (t - \tau) - \sum_{j=1}^{1, n_p} \left( 1 + (2h-j) \right) - \frac{p - (1 - \alpha) / 2b}{2b}
\]
\[
\times \exp \left\{ -c_2 \left( 1 - m \right) \rho_1(t, x_1, \tau, \xi_1) + 2^{2m} \rho_1(t, x_1, \tau, \xi_1) + \sum_{j=2}^p \rho_j (t, x_{(j)}, \tau, \xi_{(j)}) \right\} + \sum_{k=1}^{\infty} A_0 \left( \frac{\alpha^*}{2b} \right) FA_0^k (t - \tau) - \frac{p - (1 - \alpha) / 2b}{2b}
\]
\[
\times \exp \left\{ -c_5 \left( 4l+k+1 \right) \rho_1(t, x_1, \tau, \xi_1) + 2^{2p(l+k-1)} \rho_1(t, x_1, \tau, \xi_1) + \sum_{j=2}^p \rho_j (t, x_{(j)}, \tau, \xi_{(j)}) \right\},
\]
where \( l = \left\lfloor \frac{1}{\alpha^*} \right\rfloor + 1, \) and \( A_0, F \) are positive constants,
\[ F = \left( 2 \int_0^{\infty} \exp \left\{ -\frac{\alpha^*}{2b} \right\} d\alpha \right)^{n_0} \]
The series \( \sum_{m=1}^{\infty} \partial_{x_{p}} K_{m}(t, x; \tau, \xi) \) at \( 0 < \delta \leq t - \tau \leq T \) is convergent uniformly and absolutely. Then \( \partial_{x_{p}} \varphi(t, x; \tau, \xi) = \sum_{m=1}^{\infty} \partial_{x_{p}} K(t, x; \tau, \xi) \) and \( \partial_{x_{p}} K_{m}(t, x; \tau, \xi) \) are continuous, then in the domain of convergence and \( \partial_{x_{p}} \varphi(t, x; \tau, \xi) \) continuous function. Inequality (25) will be written in the form

\[
\left| \partial_{x_{p}} \varphi(t, x; \tau, \xi) \right| \leq A(t - \tau) - \sum_{j=1}^{p} \frac{(1+2b(j-1))n_{j} - p - (1-n)^{+}/2b}{2b} - \Phi(t, x; \tau, \xi).
\]

Let us consider \( \partial_{x_{p}} K(t, x; \beta, \gamma), j = \frac{2p - 1}{2}, \mu = \frac{n_{j - 1} + 1}{2}, \) formula (23) is true with the corresponding replacing \( p \) by \( p \). For \( \mu = 1, n_{j - 1}, \partial_{x_{p}} K(t, x; \beta, \gamma) \) can be written in the form

\[
\partial_{x_{p}} K(t, x; \beta, \gamma) = \sum_{|k|=2b} \left[ \partial_{x_{p}} a_{k}(t, x) - \partial_{y_{p}} a_{k}(t, y) \right]_{y=\gamma(t, \beta)} D_{x_{1}}^{k} Z_{0}(t, x; \beta, \gamma; \gamma(t, \beta)) - \partial_{\gamma_{j_{p}}} \left( \sum_{|k|=2b} \left[ a_{k}(t, x) - a_{k}(t, \gamma(t, \beta)) \right] D_{x_{1}}^{k} Z_{0}(t, x; \beta, \gamma; \gamma(t, \beta)) \right) + \sum_{|k|=2b} \left[ a_{k}(t, x) - a_{k}(t, \gamma(t, \beta)) \right] \partial_{\gamma_{j_{p}}} D_{x_{1}}^{k} Z_{0}(t, x; \beta, \gamma; \gamma(t, \beta)) \bigg|_{\tau=\gamma}
\]

\[
\times (-1)^{l} (t - \tau)^{p - l - 1} \sum_{|k|=2b} \partial_{y_{p}} a_{k}(t, y) \bigg|_{y=\gamma(t, \beta)} (-1)^{l} (t - \beta)^{p - l - 1} (p - j - l)! \times D_{x_{1}}^{k} \partial_{\gamma_{j_{p}}} Z_{0}(t, x; \beta, \gamma; \gamma(t, \beta)) + \sum_{|k|=2b} \left( a_{k}(t, x) \right)_{x_{j_{p}}} D_{x_{1}}^{k} Z_{0}(t, x; \beta, \gamma; \gamma(t, \beta))
\]

\[
+ \sum_{|k|=2b} \partial_{\gamma_{j_{p}}} a_{k}(t, x) D_{x_{1}}^{k} Z_{0}(t, x; \beta, \gamma; \gamma(t, \beta)) + \sum_{|k|=2b} a_{k}(t, x) D_{x_{1}}^{k} \partial_{\gamma_{j_{p}}} Z_{0}(t, x; \beta, \gamma; \gamma(t, \beta)) \bigg|_{\tau=\gamma}
\]

\[
+ (-1)^{l} (t - \beta)^{p - l - 1} (p - j - l)! \sum_{j=1}^{p-j} \sum_{|k|=2b} a_{k}(t, x) D_{x_{1}}^{k} \partial_{\gamma_{j_{p}}} Z_{0}(t, x; \beta, \gamma; \gamma(t, \beta)).
\]

Kernels have the highest singularity at the variable \( x_{p} \). Also, using (26) we have the existence of \( \partial_{x_{j}} \varphi(t, x; \tau, \xi), j = \frac{2p - 1}{2}, \) and the following estimations

\[
\left| \partial_{x_{j}} \varphi(t, x; \tau, \xi) \right| \leq A(t - \tau) - \sum_{s=1}^{p} \frac{(1+2b(j-1))n_{j} - s - n^{+}/2b}{2b} - \Phi(t, x; \tau, \xi), \quad j = \frac{2p - 1}{2}, \]

Using arguments like in [1] we can get

\[
\Delta_{h_{x_{1}}} \varphi(t, x; \tau, \xi) = \Delta_{h_{x_{1}}} K(t, x; \tau, \xi) + \int_{\tau}^{t} \int_{R_{0}} \Delta_{h_{x_{1}}} K(t, x; \beta, \gamma) K(\beta, \gamma; \tau, \xi) d\gamma.
\]
Applying the technique developed for parabolic systems in [6], and the evaluation of reproducing kernels, we obtain

\[ |\Delta h_{x_1} \varphi(t, x; \tau, \xi)| \leq |h_{x_1}|^{\alpha_1} (t - \tau)^{-\frac{p}{2} \sum_{s=1}^{p} \frac{(1+2b(s-1))\alpha_2 - (1-\alpha_2)}{2b}} \Phi(t, x; \tau, \xi), \]

\[ \alpha_1 > 0, \ \alpha_2 > 0, \ \alpha_1 + \alpha_1 = \alpha. \]

The existence and evaluation of \( \frac{\partial}{\partial k_{x_1}} Z(t, x; \tau, \xi), \ |k| \leq 2b, \) at \( t > \tau, \) are established for both of the cases of parabolic equations and systems in [6]. The theorem is proved.

**References**


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