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A NOTE ON APPROXIMATION OF CONTINUOUS FUNCTIONS ON NORMED SPACES 

Let $X$ be a real separable normed space $X$ admitting a separating polynomial. We prove that each continuous function from a subset $A$ of $X$ to a real Banach space can be uniformly approximated by restrictions to $A$ of functions, which are analytic on open subsets of $X$. Also we prove that each continuous function to a complex Banach space from a complex separable normed space, admitting a separating $*$-polynomial, can be uniformly approximated by $*$-analytic functions.

Key words and phrases: normed space, continuous function, analytic function, $*$-analytic function, uniform approximation, separating polynomial.

The first known result on uniform approximation of continuous functions was obtained by Weierstrass in 1885. Namely, he showed that any continuous real-valued function on a compact subset $K$ of a finitely dimensional real Euclidean space $X$ can be uniformly approximated by restrictions on $K$ of polynomials on $X$. For a compact subset $K$ of a finite dimensional complex Euclidean space $X$ holds a counterpart of Stone-Weierstrass’ theorem, according to which any continuous complex-valued function on $K$ can be approximated by elements of any algebra, containing restrictions on $K$ of polynomials on $X$ and their conjugated functions. A general direction of investigations is to try to extend these results to topological linear spaces. Most of the obtained results concern separable Banach spaces, although in the paper [4] the authors obtained partial positive results for separable Fréchet spaces. A negative result belongs to Nemirovskii and Semenov, who in [7] built a continuous real-valued function on the unit ball $K$ of the real space $\ell_2$, which cannot be uniformly approximated by restrictions onto $K$ of polynomials on $\ell_2$. This result showed that in order to uniformly approximate continuous functions on Banach spaces we need a bigger class of functions than polynomials. The following fundamental result was obtained by Kurzweil [3].

**Theorem 1.** Let $X$ be any separable real Banach space that admits a separating polynomial, $G$ be any open subset of $X$, and $F$ be any continuous map from $G$ to any real Banach space $Y$. Then for any $\varepsilon > 0$ there exists an analytic map $H$ from $G$ to $Y$ such that $\|F(x) - H(x)\| < \varepsilon$ for all $x \in G$.

Separating polynomials were introduced in [3] and are considered in reviews [2] and [6]. In order to define them and to obtain a counterpart of Kurzweil’s Theorem for a complex Banach space $X$, in paper [5] were introduced notions, which we adapt below for complex normed spaces $X$ and $Y$.

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A map $B_{km}$ from $X^{k+m}$ to $Y$ is a map of type $(k, m)$ if $B_{km}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+m})$ is a nonzero map, which is $k$-linear with respect to $x_i$, $1 \leq i \leq k$, and $m$-antilinear with respect to $x_{k+j}$, $1 \leq j \leq m$.

**Definition 1.** A map $B_n : X^n \to Y$ is $*$-n-linear if

$$B_n(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+m}) = \sum_{k+m=n} c_{km}B_{km}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+m}),$$

where for each $k$ and $m$ such that $k + m = n$, $B_{km}$ is a map of type $(k, m)$ and $c_{km}$ is either 0 or 1, and at least one of $c_{km}$ is non-zero.

**Definition 2.** A map $F_n : X \to Y$ is called an $n$-homogeneous $*$-polynomial if there exists a $*$-n-linear map $B_n : X^n \to Y$ such that $F_n(x) = B_n(x, \ldots, x)$ for all $x \in X$. Remark that $F_0$ is a constant map.

**Definition 3.** A map $F : X \to Y$ is a $*$-polynomial of degree $j$, if

$$F = \sum_{n=0}^{j} F_n,$$

where $F_n$ is an $n$-homogeneous continuous $*$-polynomial for each $n$ and $F_j \neq 0$.

**Definition 4.** A map $H : X \to Y$ is $*$-analytic if every point $x \in X$ has a neighborhood $V$ such that

$$H(x) = \sum_{n=0}^{\infty} F_n(x),$$

where for each $n$ we have that $F_n$ is an $n$-homogeneous continuous $*$-polynomial and the series $\sum_{n=0}^{\infty} F_n(x)$ converges in $V$ uniformly with respect to the norm of the space $Y$.

**Definition 5.** Let $X$ be a complex (resp. real) normed space. A $*$-polynomial (resp. polynomial) $P : X \to C$ (resp. to $\mathbb{R}$) is called a separating $*$-polynomial (resp. polynomial) if $P(0) = 0$ and $\inf_{\|x\|=1} P(x) > 0$.

Denote by $\mathcal{H}(X, Y)$ the normed space of $*$-analytic functions from $X$ to $Y$.

**Theorem 2 ([5]).** Let $X$ be any separable complex Banach space that admits a separating $*$-polynomial, $Y$ be any complex Banach space, and $F : X \to Y$ be any continuous map. Then for any $\varepsilon > 0$ there exists a map $H \in \mathcal{H}(X, Y)$ such that $\|F(x) - H(x)\| < \varepsilon$ for all $x \in X$.

The aim of the present paper is to generalize Theorems 1 and 2 to normed spaces. To this end we need the following technical result.

**Lemma 1.** If a real normed space $X$ admits a separating polynomial $q$ then its completion $\hat{X}$ admits a separating polynomial too.

**Proof.** Let $q = \sum_{i \in I} q_i$ be a sum of homogeneous polynomials $q_i$ on the space $X$. For each $i \in I$ there exists a polynlinear form $h_i : X^{n_i} \to \mathbb{R}$ such that $q_i(x) = h_i(x, \ldots, x)$ for each $x \in X$. Since $h_i$ is a Lipschitz function on $X^{n_i}$, by [1, Theorem 4.3.17], it admits a continuous extension
\(\tilde{h}_i\) on the space \(\tilde{X}^n\), which is polylinear by the polylinearity of \(h_i\). The map \(\tilde{q}_i : \tilde{X} \to \mathbb{R}\) defined as \(\tilde{q}_i(x) = h_i(x_1, \ldots, x_n)\) for each \(x \in \tilde{X}\) is an extension of the map \(q_i\). Then the map \(\tilde{q} = \sum_{i \in I} \tilde{q}_i\) is a continuous polynomial extension of the map \(q\) onto the space \(X\). It is easy to show that the unit sphere \(S\) of the space \(X\) is dense in the unit sphere \(\tilde{S}\) of the space \(\tilde{X}\). Therefore \(\inf_{x \in S} \tilde{q}(x) = \inf_{x \in S} q(x) > 0\), so \(\tilde{q}\) is a separating polynomial for the space \(\tilde{X}\).

**Theorem 3.** Let \(X\) be a separable real normed space that admits a separating polynomial, \(Y\) be a real Banach space, \(A \subset X\), \(f : A \to Y\) be a continuous function, and \(\varepsilon > 0\). Then there are an open set \(A_\varepsilon \supset A\) of \(X\) and an analytic function \(f_\varepsilon : A_\varepsilon \to Y\) such that \(\|f(x) - f_\varepsilon(x)\| < \varepsilon\) for all \(x \in A\).

**Proof.** Let \(\tilde{X}\) be a completion of \(X\). We build a cover of the set \(A\) by open in \(\tilde{X}\) sets as follows. For each point \(x \in A\) pick its neighborhood \(O(x)\) open in \(\tilde{X}\) such that \(\|f(x') - f(x)\| < \varepsilon/3\) for all \(x' \in O(x) \cap A\).

Put \(\tilde{A}_\varepsilon = \bigcup_{x \in A} O(x)\). The topological space \(\tilde{A}_\varepsilon\) is metrizable, and therefore paracompact, \([1, 5.1.3]\). Therefore, by \([1, 5.1.9]\) there is a locally finite partition \(\{\varphi_s : s \in S\}\) of the unity , subordinated to the cover \(\{O(x) : x \in A\}\).

Now we construct an auxiliary function \(f'_\varepsilon : \tilde{A}_\varepsilon \to Y\). First, for each index \(s \in S\) we define a real number \(a_s\) as follows. If \(\varphi_s \cap A \neq \emptyset\), then we pick an arbitrary point \(x_s \in \text{supp } \varphi_s \cap A\), and we put \(a_s = f(x_s)\). Otherwise, we put \(a_s = 0\). Finally, put \(f'_\varepsilon = \sum a_s \varphi_s\).

Let \(x \in A\). Put \(S_x = \{s \in S \mid x \in \text{supp } \varphi_s\}\). Then \(\sum_{s \in S_x} \varphi_s(x) = 1\). Let \(s \in S_x\) be any index. Thus there is an element \(x_0 \in A\) such that \(x \in \text{supp } \varphi_s \subset O(x_0)\). Hence \(x_s \in O(x_0)\) and

\[
\|f(x) - a_s\| = \|f(x) - f(x_s)\| \leq \|f(x) - f(x_0)\| + \|f(x_0) - f(x_s)\| < 2\varepsilon/3.
\]

Then

\[
\|f(x) - f'_\varepsilon(x)\| = \left\|f(x) - \sum_{s \in S} a_s \varphi_s(x)\right\| = \left\|\sum_{s \in S} f(x) \varphi_s(x) - \sum_{s \in S} a_s \varphi_s(x)\right\|
\]

\[
= \left\|\sum_{s \in S_x} f(x) \varphi_s(x) - \sum_{s \in S_x} a_s \varphi_s(x)\right\| \leq \sum_{s \in S_x} \|f(x) \varphi_s(x) - a_s \varphi_s(x)\|
\]

\[
= \sum_{s \in S_x} \|f(x) - a_s\| \varphi_s(x) < \sum_{s \in S_x} (2\varepsilon/3) \varphi_s(x) = 2\varepsilon/3.
\]

The function \(f'_\varepsilon\) is continuous on \(\tilde{A}_\varepsilon\) as a sum of a family of continuous functions with a locally finite family of supports.

By Lemma 1, the space \(\tilde{X}\) admits a separating polynomial. Therefore the space \(X\) satisfies the conditions of Theorem 1, so there exists a function \(\tilde{f}_\varepsilon\) analytic on \(\tilde{A}_\varepsilon\) such that \(\|\tilde{f}_\varepsilon(x) - f'_\varepsilon(x)\| < \varepsilon/3\) for all \(x \in \tilde{A}_\varepsilon\). Then for all \(x \in A\) we have

\[
\|f(x) - \tilde{f}_\varepsilon(x)\| \leq \|f(x) - f'_\varepsilon(x)\| + \|f'_\varepsilon(x) - \tilde{f}_\varepsilon(x)\| < \varepsilon.
\]

It remains to put \(A_\varepsilon = \tilde{A}_\varepsilon \cap X\) and let \(f_\varepsilon\) be the restriction of the map \(\tilde{f}_\varepsilon\) to the set \(A_\varepsilon\).
**Theorem 4.** Let $X$ be any separable complex normed space that admits a separating $\ast$-polynomial, $Y$ be any complex Banach space, and $F : X \to Y$ be any continuous map. Then for any $\varepsilon > 0$ there exists a map $H \in \mathcal{H}(X, Y)$ such that $\|F(x) - H(x)\| < \varepsilon$ for each $x \in X$.

**Proof.** The proof is almost identical to the proof of Theorem 4 from [5] with the following modifications. Instead of the application of Kurzweil’s Theorem we apply Theorem 3. Instead of [5, Lemma 2] we use the fact (proof of which is similar to that of [5, Lemma 2]) that the identity map from a complex normed space $\tilde{H}(X, Y)$ to the real normed space $\mathcal{H}(\tilde{X}, Y)$ is an isomorphism of real normed spaces. □

**References**


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Нахай $X$ є дійсним сепарабельним нормованим простором, що допускає відокремлювальний поліном. Показано, що неперевні функції з підмножини $A$ в $X$ в дійсній банахів простір можуть бути рівномірно наближені аналітичними на відкритих підмножинах $X$. Також показано, що неперевні функції у комплексний банахів простір з комплексного сепарабельного нормованого простору, що допускає відокремлювальний $\ast$-поліном, можуть бути рівномірно наближені $\ast$-аналітичними функціями.

Ключові слова і фрази: нормований простір, неперервна функція, аналітична функція, $\ast$-аналітична функція, рівномірна апроксимація, відокремлювальний поліном.