NILPOTENT LIE ALGEBRAS OF DERIVATIONS WITH THE CENTER OF SMALL CORANK

Let $K$ be a field of characteristic zero, $A$ be an integral domain over $K$ with the field of fractions $R = \text{Frac}(A)$, and $\text{Der}_K A$ be the Lie algebra of all $K$-derivations on $A$. Let $W(A) := R\text{Der}_K A$ and $L$ be a nilpotent subalgebra of rank $n$ over $R$ of the Lie algebra $W(A)$. We prove that if the center $Z = Z(L)$ is of rank $\geq n - 2$ over $R$ and $F = F(L)$ is the field of constants for $L$ in $R$, then the Lie algebra $FL$ is contained in a locally nilpotent subalgebra of $W(A)$ of rank $n$ over $R$ with a natural basis over the field $R$. It is also proved that the Lie algebra $FL$ can be isomorphically embedded (as an abstract Lie algebra) into the triangular Lie algebra $u_n(F)$, which was studied early by other authors.

Key words and phrases: derivation, vector field, Lie algebra, nilpotent algebra, integral domain.

INTRODUCTION

Let $K$ be a field of characteristic zero, $A$ be an integral domain over $K$, and $R = \text{Frac}(A)$ be its field of fractions. Recall that a $K$-derivation $D$ on $A$ is a $K$-linear operator on the vector space $A$ satisfying the Leibniz rule $D(ab) = D(a)b + aD(b)$ for any $a, b \in A$. The set $\text{Der}_K A$ of all $K$-derivations on $A$ is a Lie algebra over $K$ with the Lie bracket $[D_1, D_2] = D_1D_2 - D_2D_1$. The Lie algebra $\text{Der}_K A$ can be isomorphically embedded into the Lie algebra $\text{Der}_K R$ (any derivation $D$ on $A$ can be uniquely extended on $R$ by the rule $D(a/b) = (D(a)b - aD(b))/b^2$, $a, b \in A$). We denote by $W(A)$ the subalgebra $R\text{Der}_K A$ of the Lie algebra $\text{Der}_K R$ (note that $W(A)$ and $\text{Der}_K R$ are Lie algebras over the field $K$ but not over $R$). Nevertheless, $W(A)$ and $\text{Der}_K R$ are vector spaces over the field $R$, so one can define the rank $\text{rk}_R L$ for any subalgebra $L$ of the Lie algebra $W(A)$ by the rule $\text{rk}_R L = \dim_R L$. Every subalgebra $L$ of the Lie algebra $W(A)$ determines its field of constants in $R$ by

$$F = F(L) := \{ r \in R \mid D(r) = 0 \text{ for all } D \in L \}. $$

The product $FL = \{ \sum a_iD_i \mid a_i \in F, D_i \in L \}$ is a Lie algebra over the field $F$, this Lie algebra often has simpler structure than $L$ itself (note that such an extension of the ground field preserves the main properties of $L$ from the viewpoint of Lie theory).

We study nilpotent subalgebras $L \subseteq W(A)$ of rank $n \geq 3$ over $R$ with the center $Z = Z(L)$ of rank $\geq n - 2$ over $R$, i.e. with the center of corank $\leq 2$ over $R$. We prove that $FL$ is contained

YAK 512.5
2010 Mathematics Subject Classification: Primary 17B66; Secondary 17B05, 13N15.
in a locally nilpotent subalgebra of $W(A)$ with a natural basis over $R$, similar to the standard basis of the triangular Lie algebra $U_n(F)$ (Theorem 1). As a consequence, we get an isomorphic embedding (as Lie algebras) of the Lie algebra $FL$ over $F$ into the triangular Lie algebra $u_n(F)$ over $F$ (Theorem 2). These results generalize main results of the papers [8] and [9]. Note that the problem of classifying finite dimensional Lie algebras from Theorem 1 up to isomorphism is wild (i.e., it contains the hopeless problem of classifying pairs of square matrices up to similarity, see [3]). Triangular Lie algebras were studied in [1] and [2], they are locally nilpotent but not nilpotent.

We use standard notations. The ground field $K$ is arbitrary of characteristic zero. If $F$ is a subfield of a field $R$ and $r_1, \ldots, r_k \in R$, then $F \langle r_1, \ldots, r_k \rangle$ is the set of all linear combinations of $r_i$ with coefficients in $F$, it is a subspace in the $F$-space $R$, for an infinite set $\{r_1, \ldots, r_k, \ldots \}$ we use the notation $F \langle \{r_i\}_{i=1}^{\infty} \rangle$. The triangular subalgebra $u_n(K)$ of the Lie algebra $W_n(K) := \text{Der}_K K[x_1, \ldots, x_n]$ consists of all the derivations on $K[x_1, \ldots, x_n]$ of the form

$$D = f_1(x_2, \ldots, x_n) \frac{\partial}{\partial x_1} + \cdots + f_{n-1}(x_n) \frac{\partial}{\partial x_{n-1}} + f_n \frac{\partial}{\partial x_1},$$

where $f_i \in K[x_{i+1}, \ldots, x_n], f_n \in K$. If $D \in W(A)$, then $\text{Ker} D$ denotes the field of constants for $D$ in $R$, i.e., $\text{Ker} D = \{r \in R \mid D(r) = 0\}$.

1 Main properties of nilpotent subalgebras of $W(A)$

We often use the next relations for derivations which are well known (see, for example [7]). Let $D_1, D_2 \in W(A)$ and $a, b \in R$. Then

1) $[aD_1, bD_2] = ab[D_1, D_2] + aD_1(b)D_2 - bD_2(a)D_1$;

2) if $a, b \in \text{Ker} D_1 \cap \text{Ker} D_2$, then $[aD_1, bD_2] = ab[D_1, D_2]$.

The next two lemmas contain some results about derivations and Lie algebras of derivations.

**Lemma 1** ([6], Lemma 2). Let $L$ be a subalgebra of the Lie algebra $\text{Der}_K R$ and $F$ the field of constants for $L$ in $R$. Then $FL$ is a Lie algebra over $F$, and if $L$ is abelian, nilpotent or solvable, then so is $FL$, respectively.

**Lemma 2** ([6], Proposition 1). Let $L$ be a nilpotent subalgebra of the Lie algebra $W(A)$ with $\text{rk}_R L < \infty$ and $F = F(L)$ the field of constants for $L$ in $R$. Then

1) $FL$ is finite dimensional over $F$;

2) if $\text{rk}_R L = 1$, then $L$ is abelian and $\text{dim}_F FL = 1$;

3) if $\text{rk}_R L = 2$, then $FL$ is either abelian with $\text{dim}_F FL = 2$ or $FL$ is of the form

$$FL = F \left\langle D_2, D_1, aD_1, \ldots, \frac{a^k}{k!} D_1 \right\rangle,$$

for some $D_1, D_2 \in FL$ and $a \in R$ such that $[D_1, D_2] = 0$, $D_2(a) = 1$, $D_1(a) = 0$. 

Lemma 3. Let \( L \) be a nilpotent subalgebra of the Lie algebra \( W(A) \) of rank \( n \) over \( R \) with the center \( Z = Z(L) \) of rank \( k \) over \( R \). Then \( I := RZ \cap L \) is an abelian ideal of \( L \) with \( \text{rk}_R I = k \).

Proof. By Lemma 4 from [6], \( I \) is an ideal of the Lie algebra \( L \). Let us show that \( I \) is abelian. Let us choose an arbitrary basis \( D_1, \ldots, D_k \) of the center \( Z \) over \( R \) (i.e., a maximal by inclusion linearly independent over \( R \) subset of \( Z \)). One can easy to see that \( D_1, \ldots, D_k \) is a basis of the ideal \( I \) as well, so we can write for each element \( D \in I \)

\[
D = a_1D_1 + \cdots + a_kD_k
\]

for some \( a_1, \ldots, a_k \in R \). Since \( D_j \in Z, \ j = 1, \ldots, k \), it holds

\[
[D_j, D] = [D_j, \sum_{i=1}^{k} a_iD_i] = \sum_{i=1}^{k} D_j(a_i)D_i = 0
\]  \hspace{1cm} (1)

for \( j = 1, \ldots, k \). The derivations \( D_1, \ldots, D_n \) are linearly independent over the field \( R \), hence we obtain from (1) that \( D_j(a_i) = 0, i, j = 1, \ldots, k \). Therefore we have for each element \( \overline{D} = b_1D_1 + \cdots + b_kD_k \) of the ideal \( I \) the next equalities

\[
[D, \overline{D}] = \left[ \sum_{i=1}^{k} a_iD_i, \sum_{j=1}^{k} b_jD_j \right] = \sum_{i,j=1}^{k} a_ib_j[D_i, D_j] = 0,
\]

since \( D_i(b_j) = D_j(a_i) = 0 \) as mentioned above. The latter means that \( I \) is an abelian ideal. Besides, obviously \( \text{rk}_R I = k \). \hfill \Box

Lemma 4. Let \( L \) be a nilpotent subalgebra of the Lie algebra \( W(A) \), \( Z = Z(L) \) the center of \( L \), \( I := RZ \cap L \) and \( F \) the field of constants for \( L \) in \( R \). If for some \( D \in L \) it holds \( [D, F I] \subseteq F I \), \( [D, F I] \neq 0 \), then there exist a basis \( D_1, \ldots, D_m \) of the ideal \( F I \) of the Lie algebra \( F L \) over \( R \) and \( a \in R \) such that \( D(a) = 1, \ D_i(a) = 0, \ i = 1, \ldots, m \). Besides, each element \( \overline{D} \in F I \) of the form \( \overline{D} = f_1(a)D_1 + \cdots + f_m(a)D_m \) for some polynomials \( f_i \in F_1[t] \), where \( F_1 \) is the field of constants for the subalgebra \( L_1 = F I + FD \) in \( R \).

Proof. By Lemma 3, the intersection \( I = RZ \cap L \) is an abelian ideal of the Lie algebra \( L \) and therefore \( F I \) is an abelian ideal of the Lie algebra \( F L \). Choose a basis \( D_1, \ldots, D_m \) of \( F I \) over the field \( R \) in such a way that \( D_1, \ldots, D_m \in Z \). Then \( FZ \) is the center of the Lie algebra \( F L \). Now take any basis \( T_1, \ldots, T_s \) of the \( F \)-space \( F I \) (note that the Lie algebra \( F I \) is finite dimensional over the field \( F \) by [6]). Every basis element \( T_i \) can be written in the form \( T_i = \sum_{j=1}^{m} r_{ij}D_j, \ i = 1, \ldots, s, \) for some \( r_{ij} \in R \). Denote by \( B \) the subring \( B = F[r_{ij}, i = 1, \ldots, s, j = 1, \ldots, m] \) of the field \( R \) generated by \( F \) and the elements \( r_{ij} \). Since the linear operator \( \text{ad} D \) is nilpotent on the \( F \)-space \( F I \) the derivation \( D \) is locally nilpotent on the ring \( B \). Indeed,

\[
[D, T_i] = [D, \sum_{j=1}^{m} r_{ij}D_j] = \sum_{j=1}^{m} D(r_{ij})D_j
\]

and therefore

\[
(\text{ad} D)^{k_i}(T_i) = \sum_{j=1}^{m} D^{k_i}(r_{ij})D_j = 0
\]
for some natural $k_i$, $i = 1, \ldots, s$. Denoting $\overline{k} = \max_{1 \leq t \leq s} k_t$, we get $D^\overline{k}(r_{ij}) = 0$ and therefore $D$ is locally nilpotent on $B$. One can easily show that there exists an element $p \in B$ (a preslice) such that $D(p) \in \text{Ker} D$, $D(p) \neq 0$. Then denoting $a := p/D(p)$, we have $D(a) = 1$ (such an element $a$ is called a slice for $D$). The ring $B$ is contained in the localization $B[c^{-1}]$, where $c := D(p)$ and the derivation $D$ is locally nilpotent on $B[c^{-1}]$. Note that $B[c^{-1}] \subseteq F_1$, where $F_1$ is the field of constants for $L_1 = FI + FD$ in $R$. Besides, by Principle 11 from [4] it holds $B[c^{-1}] = B_0[a]$, where $B_0$ is the kernel of $D$ in $B[c^{-1}]$. This completes the proof because $B \subseteq B[c^{-1}]$ and every element $D$ of the quotient algebra $FI$ is of the form $D = b_1D_1 + \ldots + b_mD_m, b_i \in B$. \hfill \Box

**Lemma 5.** Let $L$ be a nilpotent subalgebra of the Lie algebra $W(A)$, $Z = Z(L)$ the center of $L$, $F$ the field of constants of $L$ in $R$ and $I = RZ \cap L$. Let $\text{rk}_R Z = n - 2$. Then the following statements for the Lie algebra $FL/FI$ hold

1) if $FL/FI$ is abelian, then $\dim_F FL/FI = 2$;

2) if $FL/FI$ is nonabelian, then there exist elements $D_{n-1}, D_n \in FL, b \in R$ such that

$$FL/FI = F \left< D_{n-1} + FI, bD_{n-1} + FI, \ldots , \frac{b^k}{k!} D_{n-1} + FI, D_n + FI \right>$$

with $k \geq 1, D_n(b) = 1, D_{n-1}(b) = 0, D(b) = 0$ for all $D \in FL$.

**Proof.** Let us choose a basis $D_1, \ldots, D_{n-2}$ of the center $Z$ over the field $R$ and any central ideal $FD_{n-1} + FI$ of the quotient algebra $FL/FI$. Denote the intersection $R(I + kD_{n-1}) \cap L$ by $L_1$. Then it is easy to see that $FL_1$ is an ideal of the Lie algebra $FL$ of rank $n - 1$ over $R$ and the Lie algebra $FL/FI_1$ is of dimension 1 over $F$ (by Lemma 5 from [6]). Let us choose an arbitrary element $D_0 \in FL \setminus FI_1$. Then $D_1, \ldots, D_n$ is a basis of the Lie algebra $FL$ over the field $R$.

**Case 1.** The quotient algebra $FL/FI$ is abelian. Let us show that

$$FL/FI = F \left< D_{n-1} + FI, D_n + FI \right>.$$ 

Indeed, let us take any elements $S_1 + FI, S_2 + FI$ of $FL/FI$ and write

$$S_1 = \sum_{i=1}^{n} r_i D_i, \quad S_2 = \sum_{i=1}^{n} s_i D_i, \quad r_i, s_i \in R, i, j = 1, \ldots, n.$$ 

From the equalities $[D_i, S_1] = [D_i, S_2] = 0, i = 1, \ldots, n - 2$ (recall that $D_i \in Z(L), i = 1, \ldots, n - 2$) it follows that

$$D_i(r_j) = D_i(s_j) = 0, i = 1, \ldots, n - 2, j = 1, \ldots, n.$$ 

(2)

Since $[FL, FI] \subseteq FI$ we have $[D_i, S_1], [D_i, S_2] \in FI$ for $i = n - 1, n$. Taking into account the equalities (2) we derive that

$$D_i(s_j) = D_i(r_j) = 0, i = n - 1, n, j = n - 1, n.$$ 

Therefore it holds $s_i, r_j \in F$ for $i = n - 1, n$ and the elements $D_{n-1} + FI, D_n + FI$ form a basis for the abelian Lie algebra $FL/FI$ over the field $F$.

**Case 2.** $FL/FI$ is nonabelian. Then $\dim_F FL/FI \geq 3$ because the Lie algebra $FL/FI$ is nilpotent. Let us show that the ideal $FI_1/FI$ of the Lie algebra $FL/FI$ is abelian (recall that

...
\[ I_1 = R(I + KD_{n-1}) \cap L. \] Since \( D_{n-1} + FI \) lies in the center of the quotient algebra \( FL/FI \) we have for any element \( rD_{n-1} + FI \) of the ideal \( FL/FI \) the following equality
\[
[D_{n-1} + FI, rD_{n-1} + FI] = FI.
\]

Hence \( D_{n-1}(r)D_{n-1} + FI = FI \). The last equality implies \( D_{n-1}(r) = 0 \). But then for any elements \( rD_{n-1} + FI, sD_{n-1} + FI \) of \( FL/FI \) we get
\[
[rD_{n-1} + FI, sD_{n-1} + FI] = [rD_{n-1}, sD_{n-1} + FI] = (D_{n-1}(s)r - sD_{n-1}(r))D_{n-1} + FI = FI.
\]

The latter means that \( FL/FI \) is an abelian ideal of \( FL/FI \).

Further, the nilpotent linear operator \( \text{ad}D_n \) acts on the linear space \( FL/FI \) with \( \ker(\text{ad}D_n) = FD_{n-1} + FI \). Indeed, let \( \text{ad}D_n(rD_{n-1} + FI) = FI \). Then \( [D_n, rD_{n-1}] \in FI \) and therefore \( D_n(rD_{n-1}) \in FI \). This relation implies \( D_n(r) = 0 \) and taking into account the equalities \( D_i(r) = 0, i = 1, \ldots, n - 1 \), we get that \( r \in F \) and \( \ker(\text{ad}D_n) = FD_{n-1} + FI \). It follows from this relation that the linear operator \( \text{ad}D_n \) on \( FL/FI \) has only one Jordan chain and the Jordan basis can be chosen with the first element \( D_{n-1} + FI \). Since \( \dim FL/FI \geq 2 \) (recall that \( \dim FL/FI \geq 3 \)) the chain is of length \( \geq 2 \). Let us take the second element of the Jordan chain in the form \( bD_{n-1} + FI, b \in R \). Then \( \text{ad}D_n(bD_{n-1} + FI) = D_{n-1} + FI \) and hence \( D_n(b) = 1 \). The inclusion \( [D_{n-1}, bD_{n-1}] \in FI \) implies the equality \( D_{n-1}(b) = 0 \), and analogously one can obtain \( D_i(b) = 0, i = 1, \ldots, n - 2 \).

If \( \dim FL/FI \geq 3 \) and \( cD_{n-1} + FI \) is the third element of the Jordan chain of \( \text{ad}D_n \), then repeating the above considerations we get \( D_n(c) = b \). Then the element \( \alpha = \frac{b^2}{2^2} - c \in R \) satisfies the relations \( D_{n-1}(\alpha) = D_n(\alpha) = 0 \) and \( D_i(\alpha) = 0, i = 1, \ldots, n - 2 \), since \( D_i(b) = D_i(c) = 0 \). Therefore, \( \alpha = \frac{b^2}{2^2} - c \in F \) and \( c = \frac{b^2}{2^2} + \alpha \). Since \( aD_{n-1} + FI \in \ker(\text{ad}D_n) \), we can take the third element of the Jordan chain in the form \( \frac{b^2}{2^2}D_{n-1} + FI \). Repeating the consideration one can build the needed basis of the Lie algebra \( FL/FI \).

**Lemma 6.** Let \( L \) be a nilpotent subalgebra of \( \mathcal{W}(A) \) with the center \( Z = Z(L) \) of \( R \). If \( F \) is the field of constants for \( L \) in \( R \) and \( I = RZ \cap L \). If \( S, T \) are elements of \( L \) such that \( [S, T] \in I \), the rank of the subalgebra \( L_1 \) spanned by \( I, S, T \) equals \( n \) and \( C_{FL}(FI) = FI \), then there exist elements \( a,b \in R \) such that \( S(a) = T(a) = 0, S(b) = 0, T(b) = 1 \) and \( D(a) = D(b) = 0 \) for each \( D \in I \). Besides, every element \( D \in FI \) can be written in the form \( D = f_1(a,b)D_1 + \cdots + f_{n-2}(a,b)D_{n-2} \) with some polynomials \( f_i(u,v) \in F[u,v] \).

**Proof.** Let us choose a basis \( D_1, \ldots, D_{n-2} \) of \( Z \) over \( R \). By the lemma conditions, one can easily see that \( D_1, \ldots, D_{n-2}, S, T \) is a basis of \( L \) over \( R \). The ideal \( FI \) of the Lie algebra \( FL \) is abelian by Lemma 3 and \( \text{ad}S, \text{ad}T \) are commuting linear operators on the vector space \( FI \) (over \( F \)). Take a basis \( T_1, \ldots, T_s \) of \( F \) over \( F \) (recall that \( \dim_F FL < \infty \) by Theorem 1 from [6]) and write
\[
T_i = \sum_{j=1}^{n-2} r_{ij} D_j \text{ for some } r_{ij} \in R, i = 1, \ldots, s, j = 1, \ldots, n - 2.
\]
Denote by
\[
B = F[r_{ij}, i = 1, \ldots, s, j = 1, \ldots, n - 2],
\]
the subring of \( R \) generated by \( F \) and all the coefficients \( r_{ij} \). Then \( B \) is invariant under the derivations \( S \) and \( T \), these derivations are locally nilpotent on \( B \) and linearly independent over \( R \) (by
the condition $C_{FL}(FI) = FI$ of the lemma). By Lemma 4, there exists an element $a \in B[c^{-1}]$ such that

$$S(a) = 1, \quad D_i(a) = 0, \quad i = 1, \ldots, n - 2,$$

(here $c = S(p)$ for a preslice $p$ for $S$ in $B$). Since $c \in \ker S$ and $[S, T] = 0$ one can assume without loss of generality that $T(c) \in \ker T$. But then $T$ is a locally nilpotent derivation on the subring $B[c^{-1}]$. Repeating these considerations we can find an element $b \in B[c^{-1}][d^{-1}]$ with $T(b) = 1$ (here $d$ is a preslice for the derivation $T$ in $B[c^{-1}]$). Denote $B_1 = B[c^{-1}, d^{-1}]$, the subring of $R$ generated by $B, c^{-1}, d^{-1}$. Then using standard facts about locally nilpotent derivations (see, for example Principle 11 in [4]) one can show that $B_1 = B_0[a, b]$, where $B_0 = \ker S \cap \ker T$. Therefore every element $h$ of $B_1$ can be written in the form $h = f(a, b)$ with $f(u, v) \in F[u, v]$. Note that

$$F = \ker T \cap \ker S \cap \bigcap_{i=1}^{n-2} \ker D_i.$$

It follows from this representation of elements of $B_1$ that every element of the ideal $FI$ can be written in the form

$$D = f_1(a, b)D_1 + \cdots + f_{n-2}(a, b)D_{n-2}$$

with some polynomials $f_i(u, v) \in F[u, v]$.

\[ \Box \]

2 \hspace{1em} THE MAIN RESULTS

**Theorem 1.** Let $L$ be a nilpotent subalgebra of rank $n \geq 3$ over $R$ from the Lie algebra $W(A)$, $Z = Z(L)$ the center of $L$ with $\rk_R Z \geq n - 2$, $F$ the field of constants of $L$ in $R$. Then one of the following statements holds:

1) $\dim_F FL = n$ and $FL$ is either abelian or is a direct sum of a nonabelian nilpotent Lie algebra of dimension 3 and an abelian Lie algebra;

2) $\dim_F FL \geq n + 1$ and $FL$ lies in one of the locally nilpotent subalgebras $L_1, L_2$ of $W(A)$ of rank $n$ over $R$, which have a basis $D_1, \ldots, D_n$ over $R$ satisfying the relations $[D_i, D_j] = 0, \quad i, j = 1, \ldots, n$, and are one of the form

$$L_1 = F \left\langle \left\{ \frac{b^i}{i!} D_1 \right\}_{i=0}^{\infty}, \ldots, \left\{ \frac{b^i}{i!} D_{n-1} \right\}_{i=0}^{\infty}, D_n \right\rangle$$

for some $b \in R$ such that $D_i(b) = 0, \quad i = 1, \ldots, n - 1, \quad$ and $D_n(b) = 1,$

$$L_2 = F \left\langle \left\{ \frac{a^i b^j}{i! j!} D_1 \right\}_{i,j=0}^{\infty}, \ldots, \left\{ \frac{a^i b^j}{i! j!} D_{n-2} \right\}_{i,j=0}^{\infty}, \left\{ \frac{b^i}{i!} D_{n-1} \right\}_{i=0}^{\infty}, D_n \right\rangle$$

for some $a, b \in R$ such that $D_{n-1}(a) = 1, \quad D_n(a) = 0, \quad D_{n-1}(b) = 0, \quad D_n(b) = 1,$

$D_i(a) = D_i(b) = 0, \quad i = 1, \ldots, n - 2.$

**Proof.** By Lemma 3, $I = RZ \cap L$ is an abelian ideal of $L$ and therefore $FI$ is an abelian ideal of the Lie algebra $FL$ (here the Lie algebra $FL$ is considered over the field $F$). Let $\dim_F FL = n$. It is obvious that $\dim_F M = \rk_R M$ for any subalgebra $M$ of the Lie algebra $FL$, in particular $\dim_F FZ \geq n - 2$ because of conditions of the theorem. We may restrict ourselves only on
nonabelian algebras and assume $\dim_F FZ = n - 2$ (in case $\dim_F FZ \geq n - 1$ the Lie algebra $FL$ is abelian). Since $FL$ is nilpotent of nilpotency class 2, one can easily show that $FL$ is a direct sum of a nonabelian Lie algebra of dimension 3 and an abelian algebra and satisfies the condition 1) of the theorem. So, we may assume further that $\dim_F FL \geq n + 1$.

**Case 1.** $\text{rk}_R Z = n - 1$. Then $FI$ is of codimension 1 in $FL$ by Lemma 5 from [6]. Therefore $\dim_F FI \geq n$ because of $\dim_F FL \geq n + 1$ and $\dim_F FL/FI = 1$. We obtain the strong inclusion $FZ \subseteq FL$ because of $\dim_F FZ = n - 1$. Take a basis $D_1, \ldots, D_{n-1}$ of $Z$ over $R$ and an element $D_n \in FL \setminus FI$. Then $D_1, \ldots, D_n$ is a basis for $FL$ over $R$ and $[D_n, FI] \neq 0$. Using Lemma 4 one can easily show that $FL$ is contained in a subalgebra of type $L_1$ from $W(A)$.

**Case 2.** $\text{rk}_R Z = n - 2$ and $\dim_F FI = n - 2$. Then $FI = FZ$, $\dim_F FL/FI \geq 3$ and therefore by Lemma 5 the quotient algebra $FL/FI$ is of the form

$$FL/FI = F\left\langle \left\{ \frac{b^i}{i!} D_{n-1} + FI \right\}_{i=0}^k, D_n + FI \right\rangle$$

for some $k \geq 1$, $b \in R$ such that $D_n(b) = 1$, $D_{n-1}(b) = 0$ and $D(b) = 0$ for each $D \in FI$.

The $F$-space

$$J = F\left\langle \left\{ \frac{b^i}{i!} D_1 \right\}_{i=0}^\infty, \ldots, \left\{ \frac{b^i}{i!} D_{n-1} \right\}_{i=0}^\infty \right\rangle$$

is an abelian subalgebra of $W(A)$ and $[FL, J] \subseteq J$. Therefore the sum

$$J + F\left\langle \left\{ \frac{b^i}{i!} D_{n-1} \right\}_{i=0}^\infty, D_n \right\rangle$$

is a subalgebra of the Lie algebra $W(A)$. If $[D_n, D_{n-1}] \neq 0$, then taking into account the relation $[D_n, D_{n-1}] \in FI$ one can write

$$[D_n, D_{n-1}] = a_1 D_1 + \cdots + a_{n-2} D_{n-2}$$

for some $a_i \in F$ (recall that $FI = FZ$). Consider the element of $W(A)$ of the form

$$\tilde{D}_{n-1} = D_{n-1} - a_1 b D_1 - \cdots - a_{n-2} b D_{n-2}.$$  

Since $[D_n, \tilde{D}_{n-1}] = 0$, $\tilde{D}_{n-1}(b) = 0$, one can replace the element $D_{n-1}$ with the element $\tilde{D}_{n-1}$ and assume without loss of generality that $[D_n, D_{n-1}] = 0$. As a result we get the Lie algebra of the type $L_1$ from the statement of the theorem.

**Case 3.** $\text{rk}_R Z = n - 2$ and $\dim_F FI > n - 2$. First, suppose $C_{FL}(FI) = FI$. Then by Lemma 6 there are a basis $D_1, \ldots, D_{n-2}$ of the ideal $FI$ over $R$ and elements $a, b \in R$ such that

$$D_{n-1}(a) = 1, D_n(a) = 0, D_{n-1}(b) = 0, D_n(b) = 1$$

and

$$D_i(a) = D_i(b) = 0, i = 1, \ldots, n - 2,$$

and each element $D \in FI$ can be written in the form

$$D = f_1(a, b) D_1 + \cdots + f_{n-2}(a, b) D_{n-2}$$
for some polynomials \( f_i(u, v) \in F[u, v] \).

Consider the \( F \)-subspace

\[
J = F[a, b]D_1 + \cdots + F[a, b]D_{n-2}
\]

of the Lie algebra \( W(A) \). It is easy to see that \( J \) is an abelian subalgebra of \( W(A) \) and \([FL, J] \subseteq J\).

If \([D_n, D_{n-1}] = 0\), then it is obvious that the subalgebra \( FL + J \) is of type \( L_2 \) of the theorem and \( FL \subseteq L_1 \). Let \([D_n, D_{n-1}] \neq 0\).

Since \([D_n, D_{n-1}] \in FL\), it follows

\[
[D_n, D_{n-1}] = h_1(a, b)D_1 + \cdots + h_{n-2}D_{n-2}
\]

for some polynomials \( h_i(u, v) \in F[u, v] \).

Then the subalgebra \( J \) has such an element

\[
T = u_1(a, b)D_1 + \cdots + u_{n-2}(a, b)D_{n-2}
\]

that \( D_n(u_i(a, b)) = h_i(a, b), i = 1, \ldots, n - 2 \) (recall that \( D_n(a) = 0, D_n(b) = 1 \)), and hence the element \( \tilde{D}_{n-1} = D_{n-1} - T \) satisfies the equality \([D_n, T] = 0\).

Replacing \( D_{n-1} \) with \( \tilde{D}_{n-1} \) we get the needed basis of the Lie algebra \( FL + J \) and see that \( FL \) can be embedded into the Lie \( L_2 \) of \( W(A) \). So in case of \( C_{FL}(FL) = FL \) the Lie algebra \( FL \) can be isomorphically embedded into the Lie algebra of type \( L_2 \) from the statement of the theorem.

Further, suppose \( C_{FL}(FL) \neq FL \). Since \( C_{FL}(FL) \supseteq FL \) one can easily show that \( D_{n-1} \in C_{FL}(FL) \setminus FI \) (note that \( FL/FI \) has the unique minimal ideal \( FD_{n-1} + FI \)). Then \([D_{n-1}, FI] = 0\), and therefore \([D_n, FI] \neq 0\). Therefore by Lemma 4 there is an element \( c \in R \) such that

\[
D_n(c) = 1, D_{n-1}(c) = 0, D_i(c) = 0, i = 1, \ldots, n - 2.
\]

Moreover, each element of \( FI \) is of the form \( g_1(c)D_1 + \cdots + g_{n-2}(c)D_{n-2} \) for some polynomials \( g_i(u) \in F[u] \). By Lemma 5, the quotient algebra \( FL/FI \) is of the form

\[
FL/FI = F\left( \left\{ \frac{b^i}{i!}D_{n-1} + FI \right\}_{i=0}^{k}, D_n + FI \right)
\]

for some \( b \in R, k \geq 1 \) such that \( D_n(b) = 1, D_{n-1}(b) = 0 \). But then

\[
D_{n-1}(b - c) = 0, D_n(b - c) = 0, D_i(b - c) = 0,
\]

and hence \( b - c = \alpha \) for some \( \alpha \in F \). Without loss of generality we can assume \( b = c \). The locally nilpotent subalgebra

\[
L_1 = F\left( \left\{ \frac{a^i b^j}{i! j!}D_1 \right\}_{i, j=0}^{\infty}, \ldots, \left\{ \frac{a^i b^j}{i! j!}D_{n-2} \right\}_{i, j=0}^{\infty}, \left\{ \frac{b^i}{i!}D_{n-1} \right\}_{i=0}^{\infty}, D_n \right)
\]

of the Lie algebra \( W(A) \) contains \( FL \) and satisfies the conditions for the Lie algebra of type \( L_2 \) from the statement of the theorem, possibly except the condition \([D_n, D_{n-1}] = 0\).

If \([D_n, D_{n-1}] \neq 0\), then from the inclusion \([D_n, D_{n-1}] \in FI \) it follows that

\[
[D_n, D_{n-1}] = f_1(b)D_1 + \cdots + f_{n-2}(b)D_{n-2}
\]

for some polynomials \( f_i(u) \in F[u] \).

One can easily show that there is such an element

\[
\overline{D} = h_1(b)D_1 + \cdots + h_{n-2}(b)D_{n-2} \in L_1,
\]

that \([D_n, \overline{D}] = [D_n, D_{n-1}] \) (one can take antiderivations \( h_i \) for polynomials \( f_i, i = 1, \ldots, n - 2 \)). Replacing \( D_{n-1} \) with \( D_{n-1} - \overline{D} \) we get the needed basis over \( R \) of the Lie algebra \( L_2 \).
Remark 1. Any Lie algebra of dimension $n$ over $F$ can be realized as a Lie algebra of rank $n$ over $R$ by Theorem 2 from [5]. So the Lie algebra of type 1) from Theorem 1 can be chosen in any way possible.

As a corollary we get the next statement about embedding of Lie algebras of derivations.

**Theorem 2.** Let $L$ be a nilpotent subalgebra of rank $n$ over $R$ of the Lie algebra $W(A)$, $Z = Z(L)$ be the center of $L$ and $F$ be the field of constants of $L$ in $R$. If $\text{rk}_R Z \geq n - 2$, then the Lie algebra $FL$ can be isomorphically embedded (as an abstract Lie algebra) into the triangular Lie algebra $u_n(F)$.

**Proof.** First, suppose $\text{dim}_F FL = n$. If $FL$ is abelian, then $FL$ is isomorphically embeddable into the Lie algebra $u_n(F)$ because the subalgebra $F \langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \rangle$ of $u_n(F)$ is abelian of dimension $n$ over $F$. So one can assume that $FL$ is nonabelian. Then by Theorem 1, $FL = M_1 \oplus M_2$, where $M_1$ is an abelian Lie algebra of dimension $n - 3$ over $F$ and $M_2$ is nilpotent nonabelian with $\text{dim}_F M_2 = 3$. The subalgebra $H_2 = F \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \rangle$ of the Lie algebra $u_n(F)$ is obviously isomorphic to $M_2$. The abelian subalgebra $H_1 = F \langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \rangle$, $n \geq 4$, is isomorphic to the Lie algebra $M_1$. So $FL \simeq H_1 \oplus H_2$ is isomorphic to a subalgebra of $u_n(F)$. Note that $H_1 \oplus H_2$ is of rank $n$ over the field $\mathbb{K}(x_1, \ldots, x_n)$ of rational functions in $n$ variables.

Next, let $\text{dim}_F FL > n$. By Theorem 1, the Lie algebra $FL$ lies in one of the subalgebras of types $L_1$ or $L_2$. Therefore it is sufficient to show that the subalgebras $L_1, L_2$ of $W(A)$ from Theorem 1 can be isomorphically embedded into the Lie algebra $u_n(F)$. In case $L_1$, we define a mapping $\varphi$ on the basis $D_1, \ldots, D_n$, $\{\frac{\partial^i}{\partial x_i} D_i\}_1^\infty$ of $L_1$ over $R$ by the rule $\varphi(D_i) = \frac{\partial}{\partial x_i}, i = 1, \ldots, n$, $\varphi(\frac{\partial^i}{\partial x_i} D_i) = \frac{x_i^i}{l!} \frac{\partial}{\partial x_i}, i = 1, \ldots, n - 1$, and then extend it on $L_1$ by linearity. One can easily see that the mapping $\varphi$ is an isomorphic embedding of the Lie algebra $L_1$ into $u_n(F)$. Analogously, on $L_2$ we define a mapping $\psi : L_2 \to u_n(F)$ by the rule

$$
\psi(D_i) = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n,
\psi(\frac{a^i b^j}{i! j!} D_k) = \frac{x_i^{i-1} x_j^j}{i! j!} \frac{\partial}{\partial x_k}, \quad k = 1, \ldots, n - 2
$$

$$
\psi(\frac{b^j}{i!} D_{n-1}) = \frac{x_i^i}{l!} \frac{\partial}{\partial x_{n-1}}, \quad i \geq 1, j \geq 1,
$$

and further by linearity. Then $\psi$ is an isomorphic embedding of the Lie algebra $L_2$ into the Lie algebra $u_n(F)$. 

**References**


Received 01.03.2020


Нехай $K$ — поле характеристики нуль, $A$ — область цілісності над $K$ з полем часток $R = Frac(A)$, і $Der_K A$ — алгебра Лі $K$-диференціювань $A$. Нехай $W(A) := RDer_K A$ і $L$ — нільпотентна підальгебра рангу $n$ над $R$ Лі алгебри $W(A)$. Ми показуємо, що якщо центр $Z = Z(L)$ має ранг $\geq n - 2$ над $R$ і $F = F(L)$ — поле констант алгебри Лі $L$ в $R$, то алгебра Лі $FL$ міститься в локально нільпотентній підальгебрі рангу $n$ над $R$ з природнім базисом над полем $R$. Також доводиться, що Лі алгебра $FL$ може бути ізоморфно вкладена (як абстрактна Лі алгебра) в трикутну алгебру Лі $u_n(F)$, що була досліджена раніше іншими авторами.

Ключові слова і фрази: диференціювання, векторне поле, алгебра Лі, нільпотентна алгебра, область цілісності.